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Bialgebra actions, twists, and universal deformation formulas

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Abstract

We use the concept of gauge transformations of quasi-Hopf algebras to study twists of algebraic structures based on actions of a bialgebra and relate this to the theory of universal deformation formulas. We establish new universal deformation formulas which are associated to central extensions of Heisenberg Lie algebras. These formulas are generalizations of the Moyal quantization formula. © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

The purpose of this paper is to use Drinfeld's theory of gauge transformations of quasi-Hopf algebras [7] to define a general notion of "twisting" algebraic structures based on actions of a bialgebra B and to relate these twists to deformation theory. We show that certain elements of $B \otimes B$ can be used to twist the multiplication of any B-module algebra or B-module coalgebra. Moreover, we focus on a certain subcategory of left A-modules which naturally twists to a subcategory of modules over the twisted algebra and we show that these two categories are equivalent. Similar statements can be made for certain categories of comodules. When the bialgebra is $(kG)^*$ with G a finite group or monoid, the algebras to which our construction applies are the G-graded algebras $A = \bigoplus_{g \in G} A_g$ (these being the $(kG)^*$-module algebras) and the modules which twist are the graded modules. For algebras graded by an infinite group or monoid it is

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more convenient to use the "dual" of our theory which will produce twists of A as a kG-comodule algebra. In these cases, the twists we obtain are precisely the "cocycle" twists of [1].

Another important case is when the bialgebra is the universal enveloping algebra of a Lie algebra. Here, the twists which naturally arise are related to deformation theory and, in particular, "universal deformation formulas" (cf. [9, Section 6] or [3]). Our theory of twisting provides alternative proofs to the basic properties of these formulas. In particular, we bypass the cohomological arguments previously used to establish that the deformations obtained from these formulas are in fact associative. The supply of such formulas is scarce. Indeed, apart from those based on commutative bialgebras only a few isolated examples are known, even though some results of Drinfel'd [5] imply that there is such a formula associated to every (constant) unitary solution to the classical Yang–Baxter equation. We broaden the supply of universal deformation formulas here by providing the first family based on non-commutative bialgebras. More specifically, for every $n \geq 3$ we present a formula based on the universal enveloping algebra $U\mathscr{H}$ where $\mathscr{H} \subset \mathfrak{sl}(n)$ is a central extension of a Heisenberg Lie algebra. These formulas are generalizations of the "quasi-exponential" formula based on the two-dimensional solvable Lie algebra, see [2]. A natural use of a universal deformation formula based on an enveloping algebra $U\mathfrak{g}$ is to deform coordinate rings of various algebraic varieties. In particular, if an algebraic group with Lie algebra $\mathfrak{g}$ acts as automorphisms of a variety V, then the formula induces a deformation of the coordinate ring $\mathcal{O}(V)$. That is, the quantization takes place solely because of the action of the group. These quantizations are generalizations of the classical "Moyal product" [14] which, in our language, is a deformation of $\mathcal{O}(\mathbb{R}^{2n})$ obtained from a universal deformation formula associated to an abelian Lie algebra. This quantization is in the "direction" of the standard Poisson bracket on $\mathcal{O}(\mathbb{R}^{2n})$. In a similar way, a universal deformation formula provides deformations of coordinate rings in the direction of a suitable Poisson bracket. In the analytic case, if G is a Lie group which acts as diffeomorphisms of a manifold M, then a formula based on $U\mathfrak{g}$ will deform the ring $C^\infty(M)$. For actions of $\mathbb{R}^d$, Rieffel has shown more – namely that this action induces a strict deformation quantization of $C^\infty(M)$, see [15]. It is natural to ask whether similar constructions exist for actions of non-abelian Lie groups.

Finally, we give here a skew form of the "quasi-exponential" formula mentioned above. A skew form of a universal deformation formula has certain parity properties that make it more convenient in the study of $*$-products on Poisson manifolds. We do not, as of yet, have a skew form for our formula based on $U\mathscr{H}$ although, in theory, every formula based on an enveloping algebra can be skew-symmetrized.

1. Twisting and deformations

Let k be a fixed commutative ring. Throughout this paper, all algebras, coalgebras, and their respective modules and comodules will be symmetric k-modules and their tensor product over k will be denoted $\otimes$. We will generally follow [12] for basic definitions and notation about bialgebras and their actions. Suppose that $B = B(\mu_B, \Delta_B, 1_B, \varepsilon_B)$ is a k-bialgebra with multiplication $\mu_B : B \otimes B \to B$, comultiplication $\Delta_B : B \to B \otimes B$, unit 1_B, and counit $\varepsilon_B : B \to k$. If M is a left B-module and $b \in B$, then the left multiplication map sending $m \in M$ to $b \cdot m$ will be denoted $b_l : M \to M$. For a right B module N, we similarly have the right multiplication map $b_r : N \to N$. In this situation, we can make $M \otimes M$ and $N \otimes N$ into, respectively, left and right $B \otimes B$-modules via the maps $(b \otimes b')_l : M \otimes M \to M \otimes M$ and $(b \otimes b')_r : N \otimes N \to N \otimes N$ which send $m \otimes m'$ to $(b \cdot a) \otimes (b' \cdot a')$ and $n \otimes n'$ to $(n \cdot b) \otimes (n' \cdot b')$.

Definition 1.1. A left B-module A is a *left B-module algebra* if $A = A(\mu_A, 1_A)$ is a unital k-algebra such that for all $b \in B$,
(1) $b \cdot 1_A = (\varepsilon(b)) \cdot 1_A$ and
(2) $b \cdot (aa') = \sum (b_{(1)} \cdot a)(b_{(2)} \cdot a')$ where $\Delta_B(b) = \sum b_{(1)} \otimes b_{(2)}$.

Condition (2) is equivalent to commutativity of the diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu_A"] \arrow[d, "(\Delta_B(b))_l"'] & A \arrow[d, "b_l"'] \\ A \otimes A \arrow[r, "\mu_A"] & A \end{tikzcd}$$

for all $b \in B$. It is also easy to see that if A is a left B-module algebra then the primitive elements of B act as derivations of A and the group-like elements act as automorphisms. In a similar way, we can define the notion of a right B-module algebra. The foregoing may be easily dualized to obtain the notion of a left or right B-module coalgebra.

Definition 1.2. An element $F \in B \otimes B$ is a *twisting element* (based on B) if
(1) $(\varepsilon_B \otimes \mathrm{Id})F = 1 \otimes 1 = (\mathrm{Id} \otimes \varepsilon_B)F$, and
(2) $[(\Delta_B \otimes \mathrm{Id})(F)](F \otimes 1) = [(\mathrm{Id} \otimes \Delta_B)(F)](1 \otimes F)$.

Note that Definition 1.2(2) is an expression which must hold in $B \otimes B \otimes B$. As we shall see, such F can be used to "twist" the multiplication of any left B-module algebra A and the comultiplication of any right B-module coalgebra C. More precisely, a twisting element provides a new multiplication on the underlying k-module of a B-module algebra and a new comultiplication on the underlying k-module of a B-module coalgebra. For the algebra case, the twisted multiplication is defined to be the composite $\mu_A \circ F_l : A \otimes A \to A$ and for the coalgebra case the twisted comultiplication is $F_r \circ \Delta_C : C \to C \otimes C$. Our discussion of twisting is inspired by the notion of "gauge transformations" which naturally arise in the theory of quasi-Hopf algebras [7]

Theorem 1.3. *Let $F \in B \otimes B$ be a twisting element.*

(1) *If A is a left B-module algebra, then $A_F = A(\mu_A \circ F_l, 1_A)$ is an associative k-algebra.*

(2) *If C is a right B-module coalgebra, then $C_F = C(F_r \circ \Delta_C, \varepsilon_C)$ is a coassociative k-coalgebra.*

Proof. We only prove (1) since its dual establishes (2). The associativity of $\mu_A \circ F_l$ can be determined by considering the following diagram:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "F_l \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes F_l"'] & A \otimes A \otimes A \arrow[r, "\mu_A \otimes \mathrm{Id}"] \arrow[d, "((\Delta_B \otimes \mathrm{Id})F)_l"'] & A \otimes A \arrow[d, "F_l"'] \\
A \otimes A \otimes A \arrow[r, "((\mathrm{Id} \otimes \Delta_B)F)_l"] \arrow[d, "\mathrm{Id} \otimes \mu_A"'] & A \otimes A \otimes A \arrow[r, "\mu_A \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \mu_A"'] & A \otimes A \arrow[d, "\mu_A"'] \\
A \otimes A \arrow[r, "F_l"] & A \otimes A \arrow[r, "\mu_A"] & A.
\end{tikzcd}$$

Each of the four inner squares commutes: the lower right square represents the associativity of μ_A, the top left square commutes by Definition 1.2(2) since we are assuming that F is a twisting element, and the commutativity of the two "off-diagonal" squares follows from Definition 1.1(2) since A is a left B-module algebra. Consequently, the outer square commutes. Now the composite of the far right column with the top row is the map $(\mu_A \circ F_l)((\mu_A \circ F_l) \otimes \mathrm{Id})$ while the composite of the bottom row with the far left column is $(\mu_A \circ F_l)(\mathrm{Id} \otimes (\mu_A \circ F_l))$ and thus $\mu_A \circ F_l : A \otimes A \to A$ is associative. To see that 1_A remains the unit under this new multiplication, note that by Definitions 1.1(1) and 1.2(1) we have $[\mu_A \circ F](1_A \otimes a) = [\mu_A \circ (\varepsilon_B \otimes \mathrm{Id})F](1_A \otimes a) = \mu_A(1_A \otimes a) = a$. Similarly, 1_A also serves as the right unit under $\mu_A \circ F$. $\square$

Remark 1.4. (1) In the same manner, if $P \in B \otimes B$ satisfies

$$(P \otimes 1)[(\Delta_B \otimes \mathrm{Id})(P)] = (1 \otimes P)[(\mathrm{Id} \otimes \Delta_B)(P)]$$

then P will twist any right B-module algebra and any left B-module coalgebra.

(2) If $A = \bigoplus_{g \in G} A_G$ is a graded algebra with G a finite group or monoid, (i.e. A is a left $(kG)^*$-module algebra) and F is a twisting element based on $(kG)^*$, then A_F is a cocycle twist in the sense of [1]. When G is infinite, we instead can use a dual construction and twist A as a kG-comodule algebra. The twists of graded algebras appearing in [16] are, in general, not obtainable by twisting elements.

A natural question is whether a twisting element can be used to twist bialgebras. So far, we are unaware of any general method of twisting both the multipliction and comultiplication of a bialgebra. There is, however, a notable exception: it is possible twist the comultiplication of B itself in such a way that it remains compatible with its original multiplication, unit, and counit. Moreover, this twisted bialgebra acts in a natural on the twisted algebras A_F and coalgebras C_F formed using Theorem 1.3. To establish this we use the fact that the right multiplication map $b_r : B \to B$ sending x to xb gives B the structure of a right B-module coalgebra. Similarly, B is also a left B-module coalgebra via left multiplication $b_l : B \to B$.

Theorem 1.5. *Let F be an invertible twisting element based on a bialgebra B.*
(1) *If $\Delta'_B = F_l^{-1} \circ F_r \circ \Delta_B$, then $B_F = B(\mu_B, \Delta'_B, 1_B, \varepsilon_B)$ is a k-bialgebra.*
(2) *If A is a left B-module algebra then A_F is a left B_F-module algebra.*
(3) *If C is a right B-module coalgebra then C_F is a right B_F-module coalgebra.*

Proof. By Theorem 1.3.2, we have that $F_r \circ \Delta_B : B \otimes B \to B$ is coassociative since F is a twisting element. This comultiplication, however, is generally not compatible with μ_B, but it further twists to one which is. To see this, first note that inverting both sides of

$$[(\Delta_B \otimes \mathrm{Id})F](F \otimes 1) = [(\mathrm{Id} \otimes \Delta_B)F](1 \otimes F)$$

yields

$$(F^{-1} \otimes 1)[(\Delta_B \otimes \mathrm{Id})F^{-1}] = (1 \otimes F^{-1})[(\mathrm{Id} \otimes \Delta_B)F^{-1}]. \tag{1.1}$$

Now the twisted coalgebra B (with comultiplication $F_r \circ \Delta_B$) remains a left B-module coalgebra since $(F_r \circ \Delta_B)(b) = (\Delta_B(b))F$ only involves right multiplication by F. In light of (1.1) and Remark 1.4, it follows that $\Delta'_B = F_l^{-1} \circ F_r \circ \Delta_B$ is coassociative. For $b \in B$, we have that $\Delta'_B(b)$ is just the conjugate $F^{-1}\Delta_B(b)F$ and so this comultiplication is compatible with μ_B. Moreover, since the algebra structure remains unchanged, the original counit $\varepsilon_B : B \to k$ is an algebra map and consequently, $B(\mu_B, \Delta'_B, 1_B, \varepsilon_B)$ is a bialgebra.

To prove Theorem 1.5(2), we first need to establish that A_F is actually a left B_F-module. To do so, we define the action of $b \in B_F$ on $a \in A_F$ to be just $b \cdot a$, the action given by the original B-module structure of A. This action is well-defined since A and B coincide as k-modules with A_F and B_F and the multiplications of B and B_F are identical. With this module structure, Definition 1.1(1) is automatically satisfied since the counits of B and B_F are also identical. To show that condition Definition 1.1(2) holds we need to use the twisted multiplication in A_F and twisted comultiplication in B_F. Now the map $(\mu_A \circ F_l) \circ (F^{-1}\Delta_B(b)F)_l : A \otimes A \to A$ is the same as $(\mu_A \circ (\Delta_B(b))_l) \circ F_l$. The latter map can be expressed as $(b_l \circ \mu_A) \circ F_l$ as A is a left B-module algebra. Hence we have the equality $(\mu_A \circ F_l) \circ (F^{-1}\Delta_B(b)F)_l = b_l \circ (\mu_A \circ F_l)$ and so A_F is a left B_F-module algebra.

The (dual) proof of Theorem 1.5(3) is omitted. □

If B is commutative then the twist B_F is just B itself and so in this case A_F and C_F are still, respectively, a left B-module algebra and right C-module coalgebra.

We now turn to the topic of twisting modules and comodules along with algebras and coalgebras. Only the details for the module/algebra case are given here as those for the comodule/coalgebra case are easily obtainable by dualization. If M is a left A-module and A_F is a twist of A, it is natural to ask whether M twists to a left A_F-module. This is, in general, not possible as we must take into account the action of the bialgebra B. We will be concerned with the following class of modules.

Definition 1.6. Suppose that A is a left B-module and let A-**Mod** be the category of all left A-modules. Define (B,A)-**Mod** to be the subcategory of A-**Mod** whose objects are those left A-modules M which are also left B-modules which satisfy $b\cdot(a\cdot m)=\sum(b_{(1)}\cdot a)(b_{(2)}\cdot m)$ for all $b\in B$, $a\in A$, and $m\in M$. Morphisms in (B,A)-**Mod** are k-linear maps which are simultaneously A-module and B-module maps.

If $A=\bigoplus_{g\in G}A_g$ is a G-graded algebra then $((kG)^*,A)$-**Mod** is the familiar category of graded left A-modules. These are left A-modules M with $M=\bigoplus_{g\in G}M_g$ as a k-module and $a_g\cdot m_h\in M_{gh}$ for all $a_g\in A_G$ and $m_h\in M_h$.

Now if $M\in(B,A)$-**Mod**, then it is possible to define a twist M_F of M which will be an A_F-module. Just as A_F and A coincide as k-modules, the modules M_F and M are also identical k-modules. To define this twist we use the fact that $M\otimes A$ is a left $B\otimes B$-module since A and M are both left B-modules. In particular, for $F=(\sum f_i\otimes f_i')\in B\otimes B$, the map $F_l:A\otimes M\to A\otimes M$ sends $(a\otimes m)$ to $\sum(f_i\cdot a)\otimes(f_i'\cdot m)$. If $\lambda:A\otimes M\to M$ is the original left A-module structure map then the twisted module structure map is $\lambda_F=\lambda\circ F_l:A_F\otimes M_F\to M_F$. It is not hard to see that M_F is a well-defined A_F-module. Indeed, the equality $\lambda_F(a\otimes\lambda_F(a'\otimes M))=\lambda_F(\mu_A\circ F_l(a\otimes a')\otimes m)$ follows from Definition 1.2(2) and the fact that $b\cdot(a\cdot m)=\sum(b_{(1)}\cdot m)(b_{(2)}\cdot a)$ while $\lambda_F(1_A\otimes m)=m$ follows from Definitions 1.1(1) and 1.2(1). Now, since the bialgebra B_F has the same algebra structure as B, the twisted module M_F is also a left B_F-module under the original action of B on M. Thus M_F is both a left A_F-module and a left B_F-module and it is not hard to verify that these actions are compatible in the sense that $M_F\in(A_F,M_F)$-**Mod**.

Theorem 1.7. *If $F\in B\otimes B$ is an invertible twisting element and A is a left B-module algebra, then the categories (B,A)-**Mod** and (B_F,A_F)-**Mod** are equivalent.*

Proof. The rule assigning each M to the twist M_F defines a functor $\mathscr{F}:(B,A)$-**Mod** $\to(B_F,A_F)$-**Mod**. This functor has a two-sided inverse because we can twist A_F and B_F back to their untwisted versions. The reason why this is possible is that F^{-1} is a twisting element based on B_F. To establish this we need to show that if $F^{-1}=\sum g_i\otimes h_i$ then

$$\left[\sum(F^{-1}\Delta_B(g_i)F)\otimes h_i\right](F^{-1}\otimes 1)=\left[\sum g_i\otimes(F^{-1}\Delta_B(h_i)F)\right](1\otimes F^{-1}).$$

This equation is equivalent to having

$$(F^{-1}\otimes 1)[(\Delta_B\otimes 1)F^{-1}]=(1\otimes F^{-1})[(1\otimes \Delta_B)F^{-1}].$$

Now the latter equation is satisfied since inverting both sides yields Definition 1.2(2) which holds since F is a twisting element based on B. We may thus form $(A_F)_{F^{-1}}$ and $(B_F)_{F^{-1}}$ which are clearly just the original A and B and, in the same way as above, we get a functor $\mathscr{F}^{-1}:(B_F,A_F)$-**Mod** $\to (B,A)$-**Mod** which is clearly a two-sided inverse to $\mathscr{F}$ and so these two categories are equivalent. □

There are remarkably few techniques to explicitly produce twisting elements. In some cases though, it is possible to multiply two twisting elements to produce a new one.

Proposition 1.8. *Suppose that B' and B'' are commuting sub-bialgebras of B and let $F\in B'\otimes B'$ and $F'\in B''\otimes B''$ be twisting elements based on B' and B'', respectively. Then FF' is a twisting element based on the smallest sub-bialgebra of B which contains B' and B''.*

Proof. Since elements of B' commute with those of B'' we have that

$$[(\Delta\otimes 1)(FF')](FF'\otimes 1)=[(\Delta\otimes 1)F](F\otimes 1)[(\Delta\otimes 1)F'](F'\otimes 1).$$

Because F and F' are UDFs, this may be expressed as

$$[(1\otimes\Delta)F](1\otimes F)[(1\otimes\Delta)F'](1\otimes F'),$$

which in turn equals

$$[(1\otimes\Delta)(FF')](1\otimes FF'). \qquad \square$$

For any bialgebra B, the element $1_B\otimes 1_B$ is obviously a twisting element; when used as in Theorems 1.3 or 1.5, it is the "identity twist", that is, it effects no change. Apart from this trivial example and the cocycle twists previously mentioned, the class of quasi-triangular bialgebras gives other twisting elements.

Definition 1.9 (*Drinfel'd* [6]). A bialgebra B is *quasi-triangular* if there is an invertible element $R=\sum a_i\otimes b_i\in B\otimes B$ such that

(1) $\Delta_B(b)=R^{(-1)}(\Delta_B^{\mathrm{op}}(b))R$ for all $b\in B$ where $\Delta_B^{\mathrm{op}}(b)=\sum b_{(2)}\otimes b_{(1)}$,

(2) $(\Delta_B\otimes \mathrm{Id})(R)=R_{13}R_{23}$, and

(3) $(\mathrm{Id}\otimes\Delta_B)(R)=R_{13}R_{12}$

where $R_{12}=R\otimes 1$, $R_{23}=1\otimes R$, and $R_{13}=\sum_i a_i\otimes 1\otimes b_i$.

A basic fact about quasi-triangular bialgebras is that the associated R satisfies the quantum Yang–Baxter equation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{1.2}$$

This fact, together with Definitions 1.9(2), (3) and Remark 1.4 imply that R^{-1} is a twisting element.

Example 1.10. (1) (Majid [11]). Let $G = \mathbb{Z}/(n\mathbb{Z})$ be the cyclic group of order n with generator g. If n is invertible in k and $q \in k$ is a primitive nth root of unity then the group bialgebra $k[G]$ then has a non-trivial quasi-triangular structure in which

$$R = \frac{1}{n} \sum_{i,j=0}^{n-1} (-1)^{i+j} q^{i+j} g^i \otimes g^j.$$

(2) (Drinfel'd [6]). If B is a finite-dimensional bialgebra then its "double" $D(B)$ is a quasi-triangular bialgebra. As a k-module, $D(B) = B^* \otimes B$; its bialgebra structure is intricate and not necessary for this paper and we refer the reader to [6] or [12] for a careful discussion of this topic.

(3) The classic example of a quasi-triangular bialgebra is the quantized enveloping algebra $U_q\mathfrak{g}$, see [6]. To obtain its quasi-triangularity, it is necessary to consider $U_q\mathfrak{g}$ as a $k[\![t]\!]$-module with $q = e^t$ and take its completion with respect to the t-adic topology. The corresponding R is then expressible as an infinite series in t.

Our main interest in the remainder of this paper will be twisting elements which can be used to obtain deformations of various algebraic structures and so we will consider those based on $k[\![t]\!]$-bialgebras. We now briefly recall the main aspect of formal algebraic deformations.

Definition 1.11. Let A be an associative k-algebra with multiplication $\mu_A : A \otimes A \to A$. Then a *formal deformation* of A is a $k[\![t]\!]$-algebra structure on the power series module $A[\![t]\!]$ with multiplication of the form

$$\mu'_A = \mu_A + t\mu_1 + t^2\mu_2 + \cdots + t^n\mu_n + \cdots,$$

where each $\mu_i : A \times A \to A$ is a k-bilinear map extended to be $k[\![t]\!]$-bilinear.

Remark 1.12. (1) Note that μ'_A is completely determined by its effect on $A \otimes A$. Even though this a formal construction it turns out that in many cases $\mu'_A(a \otimes b)$ is actually a polynomial in t and so, by specialization, we can view the deformation as a k-algebra. We can obtain the same conclusion in many cases by convergence considerations (when the base field is $\mathbb{R}$ or $\mathbb{C}$.)

(2) In a more general context, it is possible to consider deformations over other rings in the following sense: If R is a commutative ring with epimorphism $R \to k$ then a deformation of A over R is an R-algebra A' which is a flat R-module together with an isomorphism $A' \otimes_R k \cong A$. The case $R = k[q, q^{-1}]$ is of particular interest in the study of quantum groups.

Returning now to formal deformations, we say that deformations A' and A'' are equivalent if there is a $k[\![t]\!]$-algebra isomorphism $\phi : A' \to A''$ of the form $\phi = \mathrm{Id}_A +$

$t\phi_1 + t^2\phi_2 + \cdots$ where $\phi_i : A \to A$ is a k-linear map extended to be $k[\![t]\!]$-linear. If we set $\mu_0 = \mu_A$, then the associativity condition $\mu_t(\mu_t(a,b),c) = \mu_t(a,\mu_t(b,c))$ is equivalent to having

$$\sum_{\substack{i+j=n \\ i,j\geq 0}} \mu_i(\mu_j(a,b),c) - \mu_i(a,\mu_j(b,c)) = 0 \tag{1.3}$$

for all $n \geq 0$ and $a,b,c \in A$. In particular, when $n=1$ we have that the *infinitesimal*, μ_1, must satisfy

$$\mu_1(a,b)c - a\mu_1(b,c) + \mu_1(ab,c) - \mu_1(a,bc) = 0$$

and so $\mu_1 \in Z^2(A,A)$, the k-module of Hochschild 2-cocycles for the algebra A with coefficients in itself. Equivalent deformations have cohomologous infinitesimals and so $H^2(A,A)$ may be interpreted as the space of equivalence classes of infinitesimal deformations of A. Given an element of $Z^2(A,A)$, it is natural to ask whether there is a deformation of A with that infinitesimal. In general, this is not possible since there may be obstructions, all lying in $H^3(A,A)$, to "integrating" an infinitesimal to a full deformation. Namely, if $\mu + t\mu_1 + t^2\mu_2 + \cdots + t^{n-1}\mu_{n-1}$ defines an associative multiplication on $A[t]/t^n$ then

$$\sum_{\substack{i+j=n \\ i,j>0}} \mu_i(\mu_j(a,b),c) - \mu_i(a,\mu_j(b,c))$$

is automatically a Hochschild 3-cocycle and must be a coboundary if the multiplication is extendible to an associative product on $A[t]/t^{n+1}$. Even when an infinitesimal μ_1 is known to be integrable to a full deformation, e.g. whenever $H^3(A,A)=0$, very little is known on how to find $\mu_2, \mu_3, \ldots$ such that $\mu_t = \sum_{i\geq 0} t^i \mu_i$ satisfies (1.3).

It is straightforward to dualize the deformation theory of algebras to that of coalgebras. If C is a coalgebra with comultiplication $\Delta_C : C \to C \otimes C$, then a formal deformation of C is a $k[\![t]\!]$-coalgebra structure on $C[\![t]\!]$ with comultiplication of the form $\Delta'_C = \Delta_C + t\Delta_1 + \cdots + t^n\Delta_n + \cdots$ where each $\Delta_i : C \to C \otimes C$ is a k-linear map extended to be $k[\![t]\!]$-linear. Note that $\Delta'_C(c)$ does not generally lie in $C[\![t]\!] \otimes_{k[\![t]\!]} C[\![t]\!]$ and so we must consider its completion, $(C \otimes C)[\![t]\!]$ with respect to the t-adic topology. Now the coassociativity condition

$$(\Delta'_C \otimes 1)\Delta'_C = (1 \otimes \Delta'_C)\Delta'_C$$

imposes restrictions on the maps Δ_i which may be interpreted in terms of the coalgebra cohomology of C. As in the algebra case, it is generally difficult to explicitly produce deformations of a coalgebra.

One way to produce deformations for a wide class of algebras and coalgebras is by use of certain twisting elements.

Definition 1.13. A *universal deformation formula* (UDF) based on a bialgebra B is a twisting element F based on $B[\![t]\!]$ of the form

$$F = 1\otimes 1 + tF_1 + t^2F_2 + \cdots + t^nF_n + \cdots,$$

where each $F_i \in B\otimes B$.

If A is a left B-module algebra and C is a right B-module coalgebra then $A[\![t]\!]$ and $C[\![t]\!]$ naturally become a left $B[\![t]\!]$-module algebra and right $B[\![t]\!]$-module coalgebra, respectively. Now if a UDF F is based on $B[\![t]\!]$ then the twists of $A[\![t]\!]$ and $C[\![t]\!]$ obtained from Theorem 1.3 are clearly deformations of A and C. Now since any UDF is invertible, it may also be used to twist the bialgebra $B[\![t]\!]$ according to Theorem 1.5. Since the algebra structure of $B[\![t]\!]$ is strictly unchanged, this will be a "preferred deformation" of B in the sense of [9].

A fundamental question is to determine "all" UDFs based on a bialgebra B. This must be understood under a natural notion of equivalence. Namely, if F is a UDF and $f = 1 + tf_1 + \cdots + t^nf_n + \cdots \in B[\![t]\!]$ then $\overline{F} = (\Delta_B f)(F)(f^{-1}\otimes f^{-1})$ is also a UDF and we say F and $\overline{F}$ are equivalent. When used as in Theorem 1.3 or Theorem 1.5, a UDF produces equivalent deformations. Little is known about the equivalence classes of UDFs except when $B = U\mathfrak{g}$. This case is of greatest interest to us in part due to its extensive connections to the theory of quantum groups. For simplicity, we denote the comultiplication of $U\mathfrak{g}$ simply by Δ instead of $\Delta_{U\mathfrak{g}}$.

Up to equivalence, the existence problem for UDFs based on $U\mathfrak{g}[\![t]\!]$ has been completely settled by Drinfel'd in [5], (see [13] for a more detailed discussion of the ideas in [5]). Drinfel'd also gives an important connection between UDFs and solutions of the Yang–Baxter equations. Recall that $r \in \mathfrak{g}\otimes\mathfrak{g}$ is a solution to the classical Yang–Baxter equation if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

and $R \in U\mathfrak{g}\otimes U\mathfrak{g}$ is a solution to the quantum Yang–Baxter equation if it satifies (1.2). Solutions to these equations are *unitary* if $r_{12} + r_{21} = 0$ in the classical case (i.e. r is skew-symmetric) and $R_{12}R_{21} = 1$ in the quantum case.

Theorem 1.14 (Drinfel'd). *Suppose that $F = 1\otimes 1 + \sum_{i=2}^{\infty} t^iF_i$ is a UDF and let $r = F_1 - (F_1)_{21}$, the skew-symmetrization of F_1.*

(1) *r is a unitary solution to the classical Yang–Baxter equation.*

(2) *If $\overline{F}$ is a UDF equivalent to F then $F_1 - (F_1)_{21} = \overline{F}_1 - (\overline{F}_1)_{21}$.*

(3) *F is equivalent to a UDF of the form $1\otimes 1 + \frac{1}{2}tr + \sum_{i\geq 2} t^iF_i$, where $F_i = (-1)^i (F_i)_{21}$.*

(4) *$F_{21}^{-1}F$ is a unitary solution to the quantum Yang–Baxter equation.*

(5) *If S is any unitary solution to the classical Yang–Baxter equation then there is a UDF of the form $1\otimes 1 + \frac{1}{2}tS + \sum_{i=2}^{\infty} t^iS_i$.*

Thus, equivalence classes of UDFs based on $U\mathfrak{g}[\![t]\!]$ are in 1–1 correspondence with unitary solutions in $\mathfrak{g} \otimes \mathfrak{g}$ of the classical Yang–Baxter equation. Regarding Theorem 1.14(5), Drinfel'd actually shows more. Namely, he outlines a procedure which, in principle, constructs a UDF starting from a unitary solution to the classical Yang–Baxter equation. In practice, however, the computations necessary to find the coefficients F_i which comprise the UDF quickly become insurmountable. In the absence of other techniques to produce UDFs, it is not surprising that so few are known.

It turns out that there is generally a rich supply of $U\mathfrak{g}$-module algebras. First, it is easy to check that any algebra A which admits an action of $\mathfrak{g}$ as derivations is a $U\mathfrak{g}$-module algebra. A natural source of this phenomenon comes from group actions. Suppose G is an algebraic which acts as automorphisms a variety V. Then $\mathfrak{g} = Lie(G)$ acts as derivations of the coordinate ring $\mathcal{O}(V)$ of polynomial functions on V. Thus whenever we have a UDF based on $U\mathfrak{g}[\![t]\!]$ the function ring $\mathcal{O}(V)$ will deform. Now there is a special type of deformation of $\mathcal{O}(V)$ called a "star-product" ($*$-product) with respect to a Poisson bracket, cf. [10]. Recall that a Poisson bracket on $\mathcal{O}(V)$ is a Lie bracket which satisfies $\{fg,h\} = f\{g,h\} + g\{f,h\}$ for all $f,g,h \in \mathcal{O}(V)$. A $*$-product is a deformation of $\mathcal{O}(V)$ in which the product $f * g = fg + \sum_{i \geq 0} \mu_i(f,g)$ satisfies the parity condition $\mu_i(f,g) = (-1)^i \mu_i(g,f)$, the "null-on-the-constants" condition $\mu_i(1,f) = 0$ for all $i \geq 1$, and is in the "direction" of the Poisson bracket in the sense that $\mu_1(f,g) = (\frac{1}{2})\{f,g\}$. It is clear that if F is a UDF which satisfies (1.17.3) is used to deform $\mathcal{O}(V)$ then $r_l(f,g)$ is a Poisson bracket and the resulting deformation is $*$-product. Hence any quantization of $\mathcal{O}(V)$ resulting from a UDF is equivalent to a $*$-product in the direction of some Poisson bracket.

2. Explicit formulas

The first UDFs we will consider are built from an abelian bialgebra B. In this case, the UDFs have a remarkably simple form. While widely used, a formal proof of the following has not appeared in the literature and since it has extensive applications we include one for completeness.

Theorem 2.1. *Suppose that $k \supset \mathbb{Q}$. If B is a commutative bialgebra and P is its space of primitive elements, then for any $r \in P \otimes P$*

$$\exp(\mathrm{tr}) = \sum_{i=0}^{\infty} \frac{t^i}{i!} r^i = 1 \otimes 1 + \mathrm{tr} + \frac{t^2}{2!} r^2 + \cdots + \frac{t^n}{n!} r^n + \cdots$$

is a UDF.

Proof. Since $\Delta_B \otimes \mathrm{Id} : B \otimes B \to B \otimes B \otimes B$ is an algebra map, we have

$$[(\Delta_B \otimes \mathrm{Id})(\exp(\mathrm{tr}))][\exp(\mathrm{tr}) \otimes 1] = \exp[(\Delta_B \otimes \mathrm{Id})(\mathrm{tr})] \cdot \exp[\mathrm{tr} \otimes 1] \tag{2.1}$$

and

$$[(\mathrm{Id}\otimes\Delta_B)(\exp(\mathrm{tr}))][1\otimes\exp(\mathrm{tr})] = \exp[(\mathrm{Id}\otimes\Delta_B)(\mathrm{tr})]\cdot\exp[1\otimes\mathrm{tr}]. \tag{2.2}$$

Now as B is abelian, (2.1) and (2.2) coincide if and only if

$$(\Delta_B\otimes\mathrm{Id})r + r\otimes 1 = (\mathrm{Id}\otimes\Delta_B)r + 1\otimes r$$

which clearly holds whenever $r\in P\otimes P$. □

Up to equivalence, we only need to consider those $r\in P\otimes P$ which are skew-symmetric. In this case, exp(tr) satisfies Theorem 1.14(3) and thus produces $*$-products when used to deform coordinate rings of algebraic varieties.

Example 2.2. (1) The classic use of the exponential deformation formula is to quantize the coordinate ring $\mathcal{O}(\mathbb{R}^{2n}) = \mathbb{R}[x_1,\dots,x_n,y_1,\dots y_n]$ of polynomial functions, (or $C^\infty(\mathbb{R}^{2n})$ in the analytic case), with respect to the canonical Poisson bracket. For $f,g\in\mathcal{O}(\mathbb{R}^{2n})$, this bracket is

$$\{f,g\} = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\right).$$

This Poisson bracket is induced from a translation action of $\mathbb{R}^{2n}$ on itself. Since the derivations $\partial/\partial x_i$ and $\partial/\partial y_i$ mutually commute, we can exponentiate the Poisson bracket to deform $\mathcal{O}(\mathbb{R}^{2n})$. If we let $f*g$ denote the deformed product, then it is easy to verify that the quantized algebra has the Heisenberg relations $x_i*x_j=x_j*x_i$, $y_i*y_j=y_j*y_i$, and $x_i*y_j-y_j*x_i=\delta_{ij}t$. Note that for any $f,g\in\mathcal{O}(\mathbb{R}^{2n})$ the product $f*g$ is a polynomial in t and so it is meaningful to specialize t to any real number. This construction also provides a deformation of $k[x_1,\dots,x_n,y_1,\dots y_n]$ over $k[t]$ whenever $k\supset\mathbb{Q}$.

(2) Consider the commuting derivations $x_i(\partial/\partial x_i)$ of $k[x_1,\dots,x_n]$. If $k\supset\mathbb{Q}$ then for any scalars p_{ij} we can exponentiate $\sum_{i<j}p_{ij}x_i(\partial/\partial x_i)\wedge x_j(\partial/\partial x_j)$ to deform the polynomial ring $k[x_1,\dots x_n]$. The new relations are $x_i*x_j=P_{ij}x_j*x_i$ where $P_{ij}=\exp(tp_{ij})$. This algebra is the quantum n-space associated with the standard multi-parameter quantization of $\mathcal{O}(SL(n))$. If $k=\mathbb{R}$ or $\mathbb{C}$ then for any f and g in $k[x_1,\dots,x_n]$, the deformed product $f*g$ converges for all $t\in k$.

(3) Our final application of Theorem 2.1 concerns the enveloping algebra $U\mathrm{gl}_2$. Since the matrix units E_{11} and E_{22} commute, Theorem 1.5 provides a preferred bialgebra deformation of $U\mathrm{gl}_2$ in which $\Delta'(a)=[\exp(-t(E_{11}\wedge E_{22})](\Delta(a))\,[\exp(t(E_{11}\wedge E_{22})]$. It is easy to check that both E_{11} and E_{22} remain primitive while

$$\Delta'(E_{12})=E_{12}\otimes L^{-1}+L\otimes E_{12}\quad\text{and}\quad\Delta'(E_{21})=E_{21}\otimes L+L^{-1}\otimes E_{21}$$

with $L=\frac{1}{2}\exp(t(E_{11}-E_{22}))$. As this is a preferred deformation, Δ' is compatible with the original multiplication of gl_2. It is interesting to compare this with the standard deformation $U_q\mathrm{gl}_2$ in which the multiplication must also be changed.

In contrast to the commutative case, very little is known about UDFs based on non-commutative bialgebras. Even for the case of enveloping algebras of non-abelian Lie algebras, only one family of UDFs is known. Most of the remainder of this paper will be devoted to this family. It is based on an abelian extension of a Heisenberg Lie algebra. Specifically, let $k \supset \mathbb{Q}$ and let $\mathscr{H} \subset \mathfrak{sl}(n)$ be the k-Lie-algebra generated by the diagonal matrices $H_i = E_{ii} - E_{i+1,i+1}$ for $1 \le i < n$ and elements of the form E_{1p} or E_{pn} for $p = 2, \dots, n-1$. The element of $\mathscr{H} \otimes \mathscr{H}$ which will serve as the infinitesimal of the UDF is

$$\frac{1}{2}\left(\sum_{i=1}^{n-1} c_i H_i\right) \otimes E_{1n} + \sum_{i=2}^{n-1} E_{1i} \otimes E_{in}, \tag{2.3}$$

where $c_1, \dots, c_{n-1}$ are scalars with $c_1 + c_{n-1} = 1$. The infinitesimal for the special case (along with an incorrect UDF) where all $c_i = \frac{1}{2}$ appears both in [9, 3]. Before presenting the correct UDF for the general case, we need some notation which will be used for its definition and proof. Let

$$E = E_{1n}, \quad H = \frac{1}{2}\left(\sum_{i=1}^{n-1} c_i H_i\right) \quad \text{and} \quad \mathscr{X} = \sum_{i=2}^{n-1} E_{1i} \otimes E_{in}.$$

Also set $c_1 = c$ and $c_{n-1} = d$ and so $c + d = 1$. In this notation, the infinitesimal (2.3) becomes $H \otimes E + \mathscr{X}$. Now for $m > 0$ define

$$H^{\langle m \rangle} = H(H+1)\cdots(H+m-1)$$

and set $H^{\langle 0 \rangle} = 1$. If $a \in k$ is any scalar, set $H_a^{\langle m \rangle} = (H+a)^{\langle m \rangle}$, i.e.

$$H_a^{\langle m \rangle} = (H+a)(H+a+1)\cdots(H+a+m-1).$$

Theorem 2.5. *Let $H_a^{\langle m \rangle}$, E, and $\mathscr{X}$ be as defined above and for each $m \ge 0$ set*

$$F_m = \sum_{i=0}^{m} \binom{m}{i} \mathscr{X}^i (H_i^{\langle m-i \rangle} \otimes E^{m-i}).$$

Then the series

$$F = \sum_{m=0}^{\infty} \frac{t^m}{m!} F_m = 1 \otimes 1 + t(H \otimes E + \mathscr{X}) + \frac{t^2}{2!}\{H(H+1) \otimes E^2 + 2\mathscr{X}(H+1) \otimes E + \mathscr{X}^2\} + \cdots$$

is a UDF based on $U\mathscr{H}[\![t]\!]$.

For $n=2$ we have $\mathscr{X}=0$ and $F_1=H\otimes E=(1/2)(E_{11}-E_{22})\otimes E_{12}$; in this case, the corresponding UDF is the "quasi-exponential" formula $\sum_{m=0}^{\infty}(t^m/m!)H^{\langle m\rangle}\otimes E^m$ of [3]. Before proving Theorem 2.5, we first need to establish some elementary formulas.

Lemma 2.6. *In the above notation, the following identities hold for all non-negative integers r and s and all scalars a and b:*

(1) $H_a^{\langle r\rangle}H_{a+r}^{\langle s\rangle}=H_a^{\langle r+s\rangle}$.

(2) $H_a^{\langle r\rangle}-H_{a-1}^{\langle r\rangle}=rH_a^{\langle r-1\rangle}$.

(3) $H_a^{\langle r\rangle}E^s=E^sH_{(a+s)}^{\langle r\rangle}$.

(4) $(E\otimes 1)\mathscr{X}=\mathscr{X}(E\otimes 1)$ *and* $(1\otimes E)\mathscr{X}=\mathscr{X}(1\otimes E)$.

(5) $\mathscr{X}_{13}\mathscr{X}_{23}-\mathscr{X}_{23}\mathscr{X}_{13}=0$ *and* $\mathscr{X}_{13}\mathscr{X}_{12}-\mathscr{X}_{12}\mathscr{X}_{13}=0$.

(6) $(1\otimes H_a^{\langle r\rangle})\mathscr{X}=\mathscr{X}(1\otimes H_{a+d}^{\langle r\rangle})$ *and* $(H_a^{\langle r\rangle}\otimes 1)\mathscr{X}=\mathscr{X}(H_{a+c}^{\langle r\rangle}\otimes 1)$.

(7) $\mathscr{X}_{23}\mathscr{X}_{12}^r-\mathscr{X}_{12}^r\mathscr{X}_{23}=r\mathscr{X}_{12}^{r-1}\mathscr{X}_{13}(1\otimes E\otimes 1)$.

(8) $(1\otimes\Delta)\mathscr{X}^r=\sum_{i=0}^r\binom{r}{i}\mathscr{X}_{12}^i\mathscr{X}_{13}^{r-i}$ *and* $(\Delta\otimes 1)\mathscr{X}^r=\sum_{i=0}^r\binom{r}{i}\mathscr{X}_{13}^i\mathscr{X}_{23}^{r-i}$.

(9) *For any scalars a and b,* $\Delta(H_a^{\langle r\rangle})=\sum_{i=0}^r\binom{r}{i}H_b^{\langle i\rangle}\otimes H_{a-b}^{\langle r-i\rangle}$.

Proof. All of these are straightforward to establish. Properties 2.6(1) and (2) are immediate and properties 2.6(3)–(6) follow from the relations among the generators of the Lie algebra $\mathscr{H}$. Finally, 2.6(7)–(9) can each be proved using induction on r. □

Lemma 2.7. *If $m\geq 0$ then for all $i\geq 0$*

$$\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}\mathscr{X}_{13}^s\mathscr{X}_{23}^{i-s}\mathscr{X}_{12}^j(H_j^{\langle m-j\rangle}\otimes E^{m-j}\otimes 1)$$

$$=\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}\mathscr{X}_{12}^j\mathscr{X}_{13}^s\mathscr{X}_{23}^{i-s}(H_{s+j}^{\langle m-j\rangle}\otimes E^{m-j}\otimes 1). \tag{2.4}$$

Proof. We use induction on i. For $i=0$, the assertion is trivial and so assume it holds for i. For $i+1$, the top line of (2.4) becomes, by Lemma 2.6(8),

$$(\mathscr{X}_{13}+\mathscr{X}_{23})\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}\mathscr{X}_{13}^s\mathscr{X}_{23}^{i-s}\mathscr{X}_{12}^j(H_j^{\langle m-j\rangle}\otimes E^{m-j}\otimes 1).$$

Now, by the inductive hypothesis, this in turn equals

$$(\mathscr{X}_{13}+\mathscr{X}_{23})\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}\mathscr{X}_{12}^j\mathscr{X}_{13}^s\mathscr{X}_{23}^{i-s}(H_{s+j}^{\langle m-j\rangle}\otimes E^{m-j}\otimes 1)$$

which, with the aid of Lemmas 2.6(4), (5), and (7) becomes

$$\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}\mathscr{X}_{12}^{j}\mathscr{X}_{13}^{s+1}\mathscr{X}_{23}^{i-s}(H_{s+j}^{\langle m-j\rangle}\otimes E^{m-j}\otimes 1)$$
$$+\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}\mathscr{X}_{12}^{j}\mathscr{X}_{13}^{s}\mathscr{X}_{23}^{i+1-s}(H_{s+j}^{\langle m-j\rangle}\otimes E^{m-j}\otimes 1)$$
$$+\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}j\mathscr{X}_{12}^{j-1}\mathscr{X}_{13}^{s+1}\mathscr{X}_{23}^{i-s}(H_{s+j}^{\langle m-j\rangle}\otimes E^{m-j+1}\otimes 1). \tag{2.5}$$

For $i+1$, the bottom line of (2.4) is

$$\sum_{s=0}^{i+1}\binom{i+1}{s}\sum_{j=0}^{m}\binom{m}{j}\mathscr{X}_{12}^{j}\mathscr{X}_{13}^{s}\mathscr{X}_{23}^{i+1-s}(H_{s+j}^{\langle m-j\rangle}\otimes E^{m-j}\otimes 1)$$

which, by the identity $\binom{i+1}{s}=\binom{i}{s-1}+\binom{i}{s}$, can be expressed as

$$\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}\mathscr{X}_{12}^{j}\mathscr{X}_{13}^{s+1}\mathscr{X}_{23}^{i-s}(H_{s+j+1}^{\langle m-j\rangle}\otimes E^{m-j}\otimes 1)$$
$$+\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}\mathscr{X}_{12}^{j}\mathscr{X}_{13}^{s}\mathscr{X}_{23}^{i+1-s}(H_{s+j}^{\langle m-j\rangle}\otimes E^{m-j}\otimes 1). \tag{2.6}$$

Subtracting (2.5) from (2.6) gives and using Lemma 2.6(2) gives

$$\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}\mathscr{X}_{12}^{j}\mathscr{X}_{13}^{s+1}\mathscr{X}_{23}^{i-s}(-(m-j))(H_{s+j+1}^{\langle m-j-1\rangle})\otimes E^{m-j}\otimes 1)$$
$$+\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}j\mathscr{X}_{12}^{j-1}\mathscr{X}_{13}^{s+1}\mathscr{X}_{23}^{i-s}(H_{s+j}^{\langle m-j\rangle}\otimes E^{m-j+1}\otimes 1)$$

which is the same as

$$\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m}\binom{m}{j}(-(m-j))\mathscr{X}_{12}^{j}\mathscr{X}_{13}^{s+1}\mathscr{X}_{23}^{i-s}(H_{s+j+1}^{\langle m-j-1\rangle})\otimes E^{m-j}\otimes 1)$$
$$+\sum_{s=0}^{i}\binom{i}{s}\sum_{j=0}^{m-1}\binom{m}{j+1}(j+1)\mathscr{X}_{12}^{j}\mathscr{X}_{13}^{s+1}\mathscr{X}_{23}^{i-s}(H_{s+j+1}^{\langle m-j-1\rangle}\otimes E^{m-j}\otimes 1). \tag{2.7}$$

Now (2.12) is equal to zero since $\binom{m}{j}(m-j)=\binom{m}{j+1}(j+1)$. $\square$

Proof of Theorem 2.5. To verify that F is a UDF, it suffices to show that

$$\sum_{p+q=m}\binom{m}{p}[(1\otimes\Delta)(F_p)](1\otimes F_q)=\sum_{p+q=m}\binom{m}{p}[(\Delta\otimes 1)(F_p)](F_q\otimes 1) \tag{2.8}$$

for all m. The left side of (2.8) is

$$\left\{\sum_{p+q=m}\binom{m}{p}\left\{\sum_{r=0}^{p}\binom{p}{r}(1\otimes\Delta)\mathscr{X}^r)(H_r^{\langle p-r\rangle}\otimes\Delta(E^{p-r})\right\}\right\}$$
$$\times\left\{\sum_{s=0}^{q}\binom{q}{s}(1\otimes\mathscr{X}^s)(1\otimes H_s^{\langle q-s\rangle}\otimes E^{q-s})\right\}.$$

which, using Lemmas 2.6(3), (4), and (8), can be expressed as

$$\sum_{*}\binom{m}{p}\binom{p}{r}\binom{r}{u}\binom{p-r}{v}\binom{q}{s}$$
$$\times\{\mathscr{X}_{12}^{u}\mathscr{X}_{13}^{r-u}\mathscr{X}_{23}^{s}(H_r^{\langle p-r\rangle}\otimes E^v H_s^{\langle q-s\rangle}\otimes E^{p+q-r-v-s})\},\tag{2.9}$$

where $\sum_*$ will indicate that the sum is being taken over

$$p+q=m,\quad 0\le r\le p,\ 0\le u\le r,\ 0\le v\le p-r,\ \text{and}\ 0\le s\le q.\tag{2.10}$$

In order to compare this with the right-hand side of (2.8), it will be convenient to re-index its summation by setting

$$u'=s,\quad s'=r-u,\quad p'=q+u+v,\quad v'=q-s,$$
$$r'=u+s,\quad\text{and}\quad q'=p-u-v.$$

It is easy to check that the inequalities in (2.10) are equivalent to having

$$p'+q'=m,\quad 0\le r'\le p',\ 0\le u'\le r',\ 0\le v'\le p'-r',\ \text{and}\ 0\le s'\le q'.$$

Hence we may rewrite (2.9) as

$$\sum_{*}\binom{m'}{p'}\binom{p'}{r'}\binom{r'}{u'}\binom{p'-r'}{v'}\binom{q'}{s'}$$
$$\times\{\mathscr{X}_{12}^{s}\mathscr{X}_{13}^{u}\mathscr{X}_{23}^{r-u}(H_{u+s}^{\langle q+v-s\rangle}\otimes E^{q-s}H_{r-u}^{\langle p-v-r\rangle}\otimes E^{p-r})\}.\tag{2.11}$$

The right-hand side of (2.8) is

$$\left\{\sum_{p+q=m}\binom{m}{p}\left\{\sum_{r=0}^{p}\binom{p}{r}((\Delta\otimes 1)\mathscr{X}^r)(\Delta H_r^{\langle p-r\rangle}\otimes E^{p-r})\right\}\right\}$$
$$\times\left\{\sum_{s=0}^{q}\binom{q}{s}(\mathscr{X}^s\otimes 1)(H_s^{\langle q-s\rangle}\otimes E^{q-s}\otimes 1)\right\}.$$

which, by Lemmas 2.6(3), (6), (8), and (9) becomes

$$\sum_{*}\binom{m}{p}\binom{p}{r}\binom{r}{u}\binom{p-r}{v}\binom{q}{s}\mathscr{X}_{13}^{u}\mathscr{X}_{23}^{r-u}\mathscr{X}_{12}^{s}$$
$$\times(H_s^{\langle q-s\rangle}H_{b+sc}^{\langle v\rangle}\otimes E^{q-s}H_{r+sd-b+q-s}^{\langle p-r-v\rangle}\otimes E^{p-r}),\tag{2.12}$$

where b is any scalar. If we set $b=u+q-sc$ and use the fact that $c+d=1$ together with Lemma 2.7 and Lemma 2.6(1), we can rewrite (2.12) as

$$\sum_{*}\binom{m}{p}\binom{p}{r}\binom{r}{u}\binom{p-r}{v}\binom{q}{s}$$
$$\times\{\mathscr{X}_{12}^{s}\mathscr{X}_{13}^{u}\mathscr{X}_{23}^{r-u}(H_{u+s}^{\langle q+v-s\rangle}\otimes E^{q-s}H_{r-u}^{\langle p-r-v\rangle}\otimes E^{p-r})\}. \tag{2.13}$$

Now (2.11) and (2.13) coincide since

$$\binom{m}{p}\binom{p}{r}\binom{r}{u}\binom{p-r}{v}\binom{q}{s}=\binom{m'}{p'}\binom{p'}{r'}\binom{r'}{u'}\binom{p'-r'}{v'}\binom{q'}{s'}$$

and so F is a UDF. $\square$

As with the exponential UDF based on commutative bialgebras, the UDF of Theorem 2.5 can be used to produce deformations of a large class of algebras and coalgebras.

Example 2.9. (1) According to remarks following Definition 1.12, the UDF of Theorem 2.5 provides a preferred bialgebra deformation of $U\mathscr{H}$ in which $\Delta'(x)=F^{-1}\Delta(x)F$ for any $x\in U\mathscr{H}$. Recall that this comultiplication is compatible with the original multiplication, unit, and counit of $U\mathscr{H}$. Since we are considering the Lie algebra $\mathscr{H}$ as a subalgebra of $\mathfrak{gl}(n)$, we can "extend" this deformation of $U\mathscr{H}$ to a (non-standard) deformation of $U\mathfrak{gl}(n)$.

(2) Let $\mathbb{H}\subset GL(n)$ be a simply connected algebraic group with Lie algebra $\mathscr{H}$. The group $\mathbb{H}$ acts in a natural way on k^n and hence $\mathscr{H}$ acts as derivations of $\mathcal{O}(k^n)=k[x_1,\dots,x_n]$ where E_{ij} acts as $x_i(\partial/\partial x_j)$. The UDF thus deforms $k[x_1,\dots,x_n]$ and it is not hard to see that the new relations become $x_1*x_n-x_n*x_1=(c_1/2)x_1*x_1$ while $x_p*x_n-x_n*x_p=\lambda_p x_1*x_p$ with $\lambda_p=(1/2)(c_p-c_{p-1})+1$ and all other $x_i*x_j-x_j*x_i=0$. Just like Example 2.2(1), this deformation has the property that if f and g are in $k[x_1,\dots,x_n]$ then the deformed product $f*g$ is a polynomial in t and so by specialization we obtain a family of k-algebras.

Even though the UDF of Theorem 2.5 is equivalent to one satisfying (1.15.3) we do not as of yet have a closed expression for such a formula. On the positive side, we do have one for the case $n=2$. In this case, the infinitesimal is $H\otimes E=(\frac{1}{2})(E_{11}-E_{22})\otimes E_{12}$ and is based on $U\mathfrak{s}[\![t]\!]$ where $\mathfrak{s}$, the Lie algebra generated by H and E, is the two-dimensional solvable Lie algebra, (recall that $[H,E]=E$). The "skew-symmetrized" formula is also based on $U\mathfrak{s}[\![t]\!]$ and has infinitesimal $E\wedge H=E\otimes H-H\otimes E$.

Theorem 2.10. *Let H and E be generators of the two-dimensional solvable Lie algebra where $[H,E]=E$. Then if*

$$F_m=\sum_{r=0}^{m}(-1)^r\binom{m}{r}E^{m-r}H^{\langle r\rangle}\otimes E^rH^{\langle m-r\rangle}$$

the series

$$F=\sum_{m=0}^{\infty}\frac{t^m}{m!}F_m=1\otimes 1+tE\wedge H$$
$$+\frac{t^2}{2!}(E^2\otimes H^{\langle 2\rangle}-2EH^{\langle 1\rangle}\otimes EH^{\langle 1\rangle}+H^{\langle 2\rangle}\otimes E^2)+\cdots$$

is a UDF based on $U\mathfrak{s}[\![t]\!]$.

Proof. The result will follow because we can reduce to the case where H and E commute and $H^{\langle i\rangle}=H^i$. With these reductions, F becomes the exponential deformation formula which, by Theorem 2.1, is a UDF. To obtain the necessary simplification, we will use the basis $\{E^iH^j\}$ of $U\mathfrak{s}$. Note that each F_m is in "normal form", that is, each tensor factor of F_m is expressed using the preferred basis. To compute $[(\Delta\otimes 1)(F)](F\otimes 1)$ we need must consider expressions of the form

$$[(\Delta\otimes 1)(E^{p-r}H^{\langle r\rangle}\otimes E^rH^{\langle p-r\rangle})](E^{q-s}H^{\langle s\rangle}\otimes E^sH^{\langle q-s\rangle}\otimes 1)$$

or, equivalently,

$$(\Delta E^{p-r}\otimes 1)(\Delta H^{\langle r\rangle}\otimes 1)(E^{q-s}H^{\langle s\rangle}\otimes E^sH^{\langle q-s\rangle}\otimes 1)(1\otimes 1\otimes E^rH^{\langle p-r\rangle}). \tag{2.14}$$

To write (2.14) in normal form it suffices to know how to do so for expressions of the form

$$(\Delta H^{\langle r\rangle})(E^{q-s}H^{\langle s\rangle}\otimes E^sH^{\langle q-s\rangle}) \tag{2.15}$$

This can be done using Lemmas 2.6(9) and (3) from which we obtain that, in normal form, (2.15) is

$$\sum_{u=0}^{r}\binom{r}{u}E^{q-s}H^{\langle s+r-u\rangle}\otimes E^sH^{\langle q-s+u\rangle}.$$

Thus, if $[(\Delta\otimes 1)(F)](F\otimes 1)$ is expressed in normal form, the result is symbolically the same as in the case where H and E commute and $H^{\langle i\rangle}=H^i$. Of course, the same is true for $[(1\otimes\Delta)(F)](1\otimes F)$ and so by Theorem 2.1, F is a UDF based on $U\mathfrak{s}[\![t]\!]$. □

We conclude this paper with some questions about universal deformation formulas.

Questions 2.11. (1) Is there a "conceptual" way to determine whether a given series is a UDF? That is, can the direct computational method of proof be avoided?

(2) Do the UDFs of Theorems 2.5 and (2.14) globalize in an analogous way that the formula of Example 2.2(1) does? The global version of this Example 2.2(1) is the theorem which says that if M is any symplectic manifold then there is a canonical $*$-product on $C^\infty(M)$, see [4] or, for a geometric proof see [8]. Locally, any such manifold looks like $\mathbb{R}^{2n}$ with the symplectic structure given in Example (2.2.1) and so the exponential formula gives the local $*$-product. The difficult part is to show that

these local deformations are compatible. Now if M is just a Poisson manifold and locally a UDF gives a deformation of $C^{\infty}(M)$, does there exist a global deformation?

(3) Are there analogs of UDFs which give strict deformation quantizations of C^*-algebras in the sense of Rieffel (see [15])?

(4) Is there a procedure which produces a UDF based on $U\mathfrak{g}$ from a constant unitary solution to the classical Yang–Baxter equation based on a Lie algebra $\mathfrak{g}$? As stated earlier, there always is such a formula but its explicit form remains a mystery.

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